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On Real and Complex-Valued Bivariate Chebyshev Polynomials

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In the real uniform approximation of the function $x^m y^n$ by the space of bivariate polynomials of total degree $m+n-1$ on the unit square, the product of monic univariate Chebyshev polynomials yields an optimal error. We exploit the fundamental Noether's theorem of algebraic curves theory to give necessary and sufficient conditions for unicity and to describe the set of optimal errors in case of nonuniqueness. Then, we extend these results to the complex approximation on biellipses. It turns out that the product of Chebyshev polynomials also provides an optimal error and that the same kind of uniqueness conditions prevail in the complex case. Yet, when nonuniqueness occurs, the characterization of the set of optimal errors presents peculiarities, compared to the real problem.

1. INTRODUCTION

The first part of this paper is concerned with the following approximation problem.

Problem A. Given arbitrary integers m, n, with $m+n \geqslant 1$, find all elements p^* of the space P_{m+n-1} of real bivariate polynomials of total degree $m+n-1$,

$$P_{m+n-1} = \left\{ p(x, y) = \sum a_{hl} x^h y^l ;\, a_{hl} \in \mathbf{R} ;\, h+l \leqslant m+n-1 \right\}$$

which best approximate the function $f(x, y) = x^m y^n$ on the unit square $U = [-1, +1]^2$ in the uniform norm

$$\|f - p^*\| = \inf\{ \|f - p\| ;\, p \in P_{m+n-1} \},$$

where, for $e = f - p$, $\|e\| = \max\{ |e(x, y)| ;\, (x, y) \in U \}$.

In the sequel, we assume $m \geqslant n$ for obvious reasons of symmetry.

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Related problems have been considered in [4, 5, 9]. In the L_1-norm, Fromm [4] proved that the product $\tilde{U}_m(x)\,\tilde{U}_n(y)$ of monic Chebyshev polynomials of the second kind is the unique optimal error $e^*(x, y)$. In the uniform norm and on the unit disk, Gearhart [5] solved the problem with the polynomial $e^*(x, y) = \tilde{U}_m(x)\,\tilde{U}_n(y) + 2^{-4}\tilde{U}_{m-2}(x)\,\tilde{U}_{n-2}(y)$ and showed that uniqueness occurs only if $n=0$ or $m=n=1$. Other optimal errors $e^*(x, y)$ have been obtained by Reimer [9] by means of a generating function.

In the uniform norm and on the unit square, Shapiro [15, p. 36] established the following result as corollary of a general theorem [14] on approximation of product functions by the space of blending functions, which has been recently extended by Haussmann and Zeller [7].

THEOREM 1. *In the uniform approximation on the unit square, of the function* $f(x, y) = x^m y^n$ *out of the space*

$$Q_{m,n} = \left\{ p(x, y) = \sum' a_{hl} x^h y^l;\ a_{hl} \in \mathbf{R} \right\},$$

where $\sum'$ *denotes summation over any finite collection of pairs of integers* (h, l) *with* $\min(h-m, l-n) < 0$, *one has*

$$\|f - p\| \geqslant \|\tilde{T}_{m,n}\|$$

in which $\tilde{T}_{m,n}(x, y)$ *is the product* $\tilde{T}_m(x)\,\tilde{T}_n(y)$ *of monic Chebyshev polynomials of the first kind.*

Using divided difference methods, Reimer [10] proved this result by an argument based upon extremal signatures [12, 15] for $Q_{m,n}$.

Since P_{m+n-1} is a subspace of $Q_{m,n}$ and $x^m y^n - \tilde{T}_m(x)\,\tilde{T}_n(y)$ belongs to P_{m+n-1}, Theorem 1 implies that $\tilde{T}_m(x)\,\tilde{T}_n(y)$ is on optimal error $e^*(x, y)$ for Problem A. The polynomial $x^m y^n - \tilde{T}_m(x)\,\tilde{T}_n(y)$ is even an element of the subspace $R_{m,n}$ of P_{m+n-1}, defined by

$$R_{m,n} = \left\{ p(x, y) = \sum a_{hl} x^h y^l;\ a_{hl} \in \mathbf{R};\ 0 \leqslant h \leqslant m,\ 0 \leqslant l \leqslant n,\ (h, l) \neq (m, n) \right\},$$

and, as shown by Ehlich and Zeller [3], it is the unique best approximation to $x^m y^n$ out of $R_{m,n}$ (see also [16] for related matters). In [13], Rivlin has obtained the associated best strong uniqueness constant.

The first contribution of this paper is to provide unicity conditions for Problem A, by exploiting Noether's theorem of algebraic plane curves theory [19], and to describe the set of best approximations in case of nonuniqueness. Then, we deal with a natural extension to the complex case, namely, Chebyshev polynomials on biellipses, that solve the following approximation problem.

Problem B. Given the integers m, n with $m+n \geqslant 1$, $m \geqslant n$, and the real number ρ with $1 < \rho < \infty$, find all elements of the space P_{m+n-1} of complex polynomials of degree $m+n-1$

$$p(w, z) = \sum a_{hl} w^h z^l, \qquad a_{hl} \in \mathbf{C}, \qquad h+l \leqslant m+n-1,$$

which best approximate $w^m z^n$ in the uniform norm on the biellipse $B_\rho = E_\rho^2$ (and therefore on the closure of the inside of B_ρ in view of the maximum modulus theorem [6, p. 7], where E_ρ is the set of complex numbers defined by

$$\tfrac{1}{2}(t + t^{-1}), \qquad |t| = \rho.$$

For the sake of simplicity, we use the same symbols for spaces of real and complex coefficient polynomials.

It is proved in a simple manner that the product of monic Chebyshev polynomials of the first kind also yields an optimal error for Problem B and that the same unicity conditions prevail in the complex case. Yet, when nonuniqueness occurs, the characterization of the set of best approximants presents peculiarities, compared with Problem A.

2. Extremal Signatures in Characterization and Uniqueness Theorems

Our arguments rely on the notion of extremal signature, which will be introduced for the complex Problem B, since it involves the real approximation problem as a particular case. A function S, defined on a finite support $D = \{(w_i, z_i);\ 1 \leqslant i \leqslant k\}$ is called a signature if $|S(w_i, z_i)| = 1$ for $i = 1, 2, ..., k$; S' is a subsignature of S if it is the restriction of S to a subset of D. The signature S is extremal for P_{m+n-1} if there exist nonzero complex numbers s_i $(1 \leqslant i \leqslant k)$, whose $\operatorname{sgn} s_i = |s_i|^{-1} s_i$ is the complex conjugate value $\overline{S(w_i, z_i)}$ of the signature at (w_i, z_i), so that

$$\sum_{i=1}^{k} s_i p(w_i, z_i) = 0, \qquad s_i \neq 0, \qquad \operatorname{sgn} s_i = \overline{S(w_i, z_i)}, \qquad \text{all} \quad p \in P_{m+n-1}, \quad (1)$$

where the coefficients s_i are normalized by $\sum_i |s_i| = 1$ with no loss of generality. The extremal signature is said to be primitive if it has no proper extremal subsignature. It is well known that extremal signatures are related to H-sets [17], but it must be stressed that all coefficients s_i in (1) are supposed to be nonzero, as required in Theorem 3. In fact, primitive extremal signatures are exact analogues of minimal H-sets.

Characterization theorems can be simply stated in terms of extremal signatures [10, 14].

THEOREM 2. *The polynomial $p^* \in P_{m+n-1}$ is a best uniform approximant of f on B_ρ iff there is an associated extremal signature S with support*

$$D = \{(w_i, z_i); i = 1, 2, ..., k\} \tag{2}$$

and relation (1), *such that D is included in the set E of extreme points $\{(w, z) \in B_\rho;\ |e^*(w, z)| = \|e^*\|;\ e^* = f - p^*\}$ with $S(w_i, z_i) = \operatorname{sgn} e^*(w_i, z_i)$.*

On the other hand, one has the following uniqueness result.

THEOREM 3. *All best approximations agree on the set D given by* (2).

Proof. The relation (1) can be written

$$\sum_{i=1}^{k} |s_i| \operatorname{sgn} \overline{e^*(w_i, z_i)}\, p(w_i, z_i) = 0, \qquad \text{all} \quad p \in P_{m+n-1}. \tag{3}$$

If $\hat{p} \in P_{m+n-1}$ denotes another best approximation of f, i.e., $\|\hat{e}\| = \|f - \hat{p}\| = \|e^*\|$, (3) becomes for $p = \hat{p} - p^* = e^* - \hat{e}$,

$$\sum_{i=1}^{k} |s_i| \operatorname{sgn} \overline{e^*(w_i, z_i)}\, e^*(w_i, z_i) = \sum_{i=1}^{k} |s_i| \operatorname{sgn} \overline{e^*(w_i, z_i)}\, \hat{e}(w_i, z_i).$$

The left side is equal to $\|e^*\|$ whereas the right side can be bounded according to

$$\sum_{i=1}^{k} |s_i| \operatorname{sgn} \overline{e^*(w_i, z_i)}\, \hat{e}(w_i, z_i) \leqslant \sum_{i=1}^{k} |s_i|\ |\hat{e}(w_i, z_i)| \leqslant \|\hat{e}\|.$$

Therefore, since $\|\hat{e}\| = \|e^*\|$ and $s_i \neq 0$ for all i, we successively deduce $\operatorname{sgn} \hat{e}(w_i, z_i) = \operatorname{sgn} e^*(w_i, z_i)$ and $|\hat{e}(w_i, z_i)| = |e^*(w_i, z_i)|$, that is, $\hat{e}(w_i, z_i) = e^*(w_i, z_i)$ or $\hat{p}(w_i, z_i) = p^*(w_i, z_i)$ for $i = 1, 2, ..., k$. ∎

An immediate consequence is the following corollary.

COROLLARY 1. *If p^* is a best approximation of f out of P_{m+n-1}, with the support* (2) *of the associated signature S, every best aproximation can be decomposed as $p^* + p$ where $p \in P_{m+n-1}$ vanishes at all points of D*

$$p \in P_{m+n-1}, \qquad p(w_i, z_i) = 0, \qquad 1 \leqslant i \leqslant k. \tag{4}$$

Of course, we get a sufficient condition for uniqueness of the best approximation if 0 is the only element of P_{m+n-1} satisfying (4): S is then called a strong extremal signature [8].

3. Problem A: The Real Case

Theorem 1 can be proved in a very concise way.

Proof of Theorem 1. Univariate Chebyshev polynomials of the first kind satisfy

$$T_m(x_k) = (-1)^k, \qquad x_k = \cos\frac{k\pi}{m}, \qquad 0 \leqslant k \leqslant m$$

$$T_n(y_j) = (-1)^j, \qquad y_j = \cos\frac{j\pi}{n}, \qquad 0 \leqslant j \leqslant n$$

so that

$$\sum_{k=0}^{m}{}'' (-1)^k x_k^h = 0, \qquad 0 \leqslant h < m$$

$$\sum_{j=0}^{n}{}'' (-1)^j y_j^l = 0, \qquad 0 \leqslant l < n,$$

where $\sum''$ indicates that the first and last terms are halved.

Consequently, we get for the bivariate polynomial $T_m(x)\,T_n(y)$, the extremal values $T_m(x_k)\,T_n(y_j) = (-1)^{k+j}$, together with the relation

$$\sum_{k=0}^{m}{}'' \sum_{j=0}^{n}{}'' (-1)^{k+j} x_k^h y_j^l = 0, \qquad \min(h-m, l-n) < 0. \tag{5}$$

Hence the set of extreme points of $T_m(x)\,T_n(y)$

$$E = \{(x_k, y_j); 0 \leqslant k \leqslant m, 0 \leqslant j \leqslant n\} \tag{6}$$

is the support of an extremal signature for $Q_{m,n}$ with the corresponding signs $(-1)^{k+j}$ and the proof is completed by virtue of Theorem 2. ∎

In fact, the extremal signature whose support is (6) is primitive for the space P_{m+n-1} of bivariate polynomials of degree $m+n-1$. In other words, E belongs to some classes of minimal H-sets for P_{m+n-1} which have been enumerated in [17]. To prove this result, we need Noether's fundamental theorem of algebraic plane curves theory [18, 19] that we give in its simplest statement.

THEOREM 4 (Noether's Theorem). *Let* $\hat{u} \in P_\mu$, $\hat{v} \in P_\nu$ *define two algebraic curves* $\hat{u}(x, y) = 0$, $\hat{v}(x, y) = 0$ *having no common component and* $\mu\, \nu$ *distinct intersection points* (x_i, y_i), $i = 1, 2, ..., \mu\, \nu$. *The curve* $p(x, y) = 0$ *related to a third polynomial* $p \in P_\sigma$, *will pass through all points* (x_i, y_i) $(1 \leqslant i \leqslant \mu\, \nu)$, *iff p has the form* $p = u\, \hat{u} + v\, \hat{v}$ *with* $u \in P_{\sigma - \mu}$, $v \in P_{\sigma - \nu}$.

With the aid of Noether's theorem, we establish

THEOREM 5. *The set E given in* (6), *is the support of a primitive extremal signature for* P_{m+n-1}.

Proof. In case of $n = 0$, E consists of $m+1$ collinear points with alternating signs, which are well-known supports of primitive extremal signatures for P_{m-1}.

For $n \neq 0$, as we deal with the real case, the theorem is true if the space

$$V = \operatorname{span}\{[1\ x\ y \cdots y^{m+n-1}] \in \mathbf{R}^{(m+n+1)(m+n)/2};\ \text{all } (x, y) \in E\}$$

has dimension card $E - 1$ [17].

If we consider the space of polynomials vanishing at the points of E

$$W = \{p \in P_{m+n-1},\ p(x, y) = 0, \text{ all } (x, y) \in E\}, \tag{7}$$

we obtain its dimension by

$$\dim W = \dim P_{m+n-1} - \dim V. \tag{8}$$

Since E is the complete intersection of the two algebraic curves

$$\hat{u}(x, y) = \prod_{k=0}^{m} (x - x_k) = 0, \qquad \hat{v}(x, y) = \prod_{j=0}^{n} (y - y_j) = 0, \tag{9}$$

Noether's theorem asserts that all elements of W can be written $p = u\, \hat{u} + v\, \hat{v}$ where u and v are arbitrary polynomials of degree $n-2$ and $m-2$, respectively. The modular law for the sum of spaces thus yields

$$\dim W = \dim P_{n-2} + \dim P_{m-2}. \tag{10}$$

Equating (8) and (10), we get

$$\begin{aligned} \dim V &= \dim P_{m+n-1} - \dim P_{n-2} - \dim P_{m-2} \\ &= (m+n+1)(m+n)/2 - n(n-1)/2 - m(m-1)/2, \end{aligned}$$

or, by an easy transformation

$$\dim V = (m+1)(n+1) - 1 = \operatorname{card} E - 1. \quad \blacksquare$$

We now turn to the question of uniqueness and prove

THEOREM 6. *The best approximation to $x^n y^n$, $m+n \geqslant 1$, $m \geqslant n$, on U out of P_{m+n-1} is unique if $n=0$ or $n=m$.*

Proof. By Corollary 1, every optimal error has the form

$$\tilde{T}_m(x)\, \tilde{T}_n(y) + p(x, y), \qquad p \in P_{m+n-1}, \tag{11}$$

where p vanishes at all points of the set E given by (6).

In case of $n=0$, E may be defined for all $y \in [-1, +1]$ as (x_k, y), $x_k = \cos k\pi/m$, $k = 0, 1, ..., m\}$. Writing $p(x, y) = \sum_{i=0}^{m-1} a_i(x)\, y^i$ where a_i has degree $m-1-i$, we get $a_i(x_k) = 0$, $k = 0, 1, ..., m$, i.e., $a_i = 0$, for $i = 0, 1, ..., m-1$. Hence, p is the zero polynomial.

For $n \neq 0$, p which belongs to the space W defined in (7) may be expressed by $p = \hat{u}\, u + \hat{v}\, v$, $u \in P_{n-2}$, $v \in P_{m-2}$, where, by (9), $\hat{u}$ and $\hat{v}$ are

$$\hat{u}(x, y) = \prod_{k=0}^{m} (x - x_k) = (x^2 - 1)\, m^{-1} \tilde{T}_m'(x),$$

$$\hat{v}(x, y) = \prod_{j=0}^{n} (y - y_j) = (y^2 - 1)\, n^{-1} \tilde{T}_n'(y).$$

All extrema arising at interior points of U must be stationary points of (11), which yields $u(x_k, y_j) = v(x_k, y_j) = 0$ for $0 < k < m$ and $0 < j < n$. Applying again Noether's theorem gives $u = 0$ and $v(x, y) = n^{-1} \tilde{T}_n'(y)$ $q(x, y)$, $q \in P_{m-n-1}$. To sum up, the best error function is of the form

$$\tilde{T}_m(x)\, \tilde{T}_n(y) + (y^2 - 1)[n^{-1} \tilde{T}_n'(y)]^2\, q(x, y), \qquad q \in P_{m-n-1}, \tag{12}$$

so that uniqueness occurs for $m = n$. ∎

To state the next theorem dealing with the nonuniqueness case, we transform (12) by means of classic relations of Chebyshev polynomials into

$$2^{-m-n+2}\{T_m(x)\, T_n(y) + [T_{2n}(y) - 1]\, r(x, y)\}, \qquad r \in P_{m-n-1} \tag{13}$$

in which r is the polynomial q with a different normalization, i.e., $r = 2^{m-n-1} q$.

THEOREM 7. *For $m > n \neq 0$, the function (13) is the error of a best approximation to $x^m y^n$ out of P_{m+n-1} if the norm $\|r\| = \max\{|r(x, y)|, (x, y) \in U\}$ of $r \in P_{m+n-1}$ obeys*

$$\|r\| \leqslant \tfrac{1}{4}. \tag{14}$$

Proof. Putting $x=\cos\phi$, $y=\cos\theta$ in (13) yields $2^{-m-n+2}g(\phi,\theta)$ with

$$g(\phi,\theta)=\cos m\phi\cos n\theta-2\sin^2 n\theta\, r(\cos\phi,\cos\theta), \tag{15}$$

and this corresponds to an optimal error iff $|g(\phi,\theta)|\leqslant 1$, all $(\phi,\theta)\in\mathbf{R}^2$.

Taking first a polynomial r of degree 0, i.e., $r(x,y)=\alpha$, and setting $\cos m\phi=a$, $\cos n\theta=b$ in $g(\phi,\theta)$, we get the function $h(a,b)=ab-2\alpha(1-b^2)$, subject to $|h(a,b)|\leqslant 1$ on the unit square U of the (a,b)-plane. On the boundaries, one has $h(a,\pm 1)=\pm a$ so that $|h(a,\pm 1)|\leqslant 1$ for $|a|\leqslant 1$, and $h(\pm 1,b)=\pm b-2\alpha(1-b^2)$ which obeys $|h(\pm 1,b)|\leqslant 1$, all $|b|\leqslant 1$, iff $|\alpha|\leqslant\frac{1}{4}$. On the other hand, the only stationary point of $h(a,b)$ is $(0,0)$ so that $h(0,0)=-2\alpha$ and $|h(0,0)|\leqslant\frac{1}{2}$ for $|\alpha|\leqslant\frac{1}{4}$. We have thus shown

$$|\cos m\phi\cos n\theta-2\sin^2 n\theta\,\alpha|\leqslant 1,\qquad \text{all } (\phi,\theta)\in\mathbf{R}^2,\qquad \text{all } |\alpha|\leqslant\tfrac{1}{4}. \tag{16}$$

If r is a nonconstant polynomial satisfying (14), one has for all $(\phi,\theta)\in\mathbf{R}^2$, $|r(\cos\phi,\cos\theta)|\leqslant\frac{1}{4}$ and, replacing in (16) α by $r(\cos\phi,\cos\theta)$, $|g(\phi,\theta)|\leqslant 1$, which completes the proof. ∎

From the above proof, it is worth emphasizing that (14) is both necessary and sufficient to ensure that (13) is an optimal error for a constant polynomial r. It is not hard to extend this result when r has degree 1 so that (13) *and* (14) *characterize the whole set of best errors in case of* $m=n+1$ *and* $m=n+2$. For $m>n+2$, this is no longer true because condition (14) is only sufficient. For instance, if we choose $r(x,y)=\alpha[T_{2n}(y)-1]$, by the same kind of procedure that was used in the proof of Theorem 7, we verify that r corresponds to an optimal error iff its norm $\|r\|$ is bounded by $\frac{27}{64}>\frac{1}{4}$.

In [10], Reimer considered the space

$$S_{m,n}=\left\{p(x,y)=\sum a_{hl}x^hy^l,\ a_{hl}\in\mathbf{R},\ h+l\leqslant m+n,\ (h,l)\neq(m,n)\right\}$$

that obeys the inclusions $P_{m+n-1}\subseteq S_{m,n}\subseteq Q_{m,n}$. By virtue of Theorem 1, $\tilde{T}_m(x)\,\tilde{T}_n(y)$ *is also the error of a best approximation on* U, *of* x^my^n *out of* $S_{m,n}$, *but, in this case, nonunicity occurs even for* $m=n$. Indeed, by the foregoing arguments, we obtain, for $m=n$, the set of best errors

$$2^{-2n+2}\{T_n(x)\,T_n(y)+\alpha[T_{2n}(x)-1]+\beta[T_{2n}(y)-1]\},\qquad |\alpha|,|\beta|\leqslant\tfrac{1}{4}. \tag{17}$$

This expression contains the particular solution obtained by Buck in [1]. He showed that, on the square $U'=\{0\leqslant X\leqslant 1, 0\leqslant Y\leqslant 1\}$, the function XY has among those polynomials of the form $p(X,Y)=$

$a_0+a_1(X+Y)+a_2(X^2+Y^2)$, infinitely many polynomials of best approximation, given by

$$\gamma f_1+\delta f_2, \qquad \gamma \geqslant 0, \quad \delta \geqslant 0, \quad \gamma+\delta=1, \tag{18}$$

where

$$f_1(X, Y)=\tfrac{1}{2}(X^2+Y^2)-\tfrac{1}{4},$$
$$f_2(X, Y)=X+Y-\tfrac{1}{2}(X^2+Y^2)-\tfrac{1}{4}.$$

Performing the change of variables $X=(x+1)/2$, $Y=(y+1)/2$, to get the unit square U of the (x, y)-plane, we can rewrite the error of (18) as

$$\tfrac{1}{4}[xy+2\alpha(x^2-1)+2\alpha(y^2-1)], \qquad |\alpha| \leqslant \tfrac{1}{4}$$

which is (17) for $n=1$ and $\alpha=\beta$, multiplied by $\frac{1}{4}$ since U' is one-fourth of the unit square U of the (X, Y)-plane.

4. Problem B: The Complex Case

It is well known [12, 15], that the normalized Chebyshev polynomial $\tilde{T}_n(z)=2^{1-n}T_n(z)$ is the monic polynomial of degree n, having the smallest uniform norm on the ellipse $E_\rho=\{z=z(t)=(t+t^{-1})/2, |t|=\rho, 1<\rho<\infty\}$ and, consequently, on the closure of the inside of E_ρ. Indeed, the set E of extreme points of $T_n(z)$ is $\{z_j=z(t_j), t_j=\rho \exp(ij\pi/n), j=0, 1, ..., 2n-1\}$ such that $T_n(z_j)=(-1)^j \|T_n\|$ with $\|T_n\|=(\rho^n+\rho^{-n})/2$. As E is the support of an extremal signature for the space of univariate polynomials of degree $n-1$, with

$$\sum_{j=0}^{2n-1} (-1)^j z_j^l=0, \qquad 0 \leqslant l \leqslant n-1,$$

the optimality of $\tilde{T}_n(z)$ follows from the characterization theorem.

In the bivariate case, we obtain an immediate equivalent to Theorem 1.

Theorem 8. *In the uniform approximation on the biellipse $B_\rho=E_\rho^2$ of the function $f(w, z)=w^m z^n$, out of the space*

$$Q_{m,n}=\left\{p(w, z)=\sum{}' a_{hl} w^h z^l, a_{hl} \in \mathbf{C}\right\},$$

where $\sum'$ denotes summation over any finite collection of pairs of integers (h, l) with $\min(h-m, l-n)<0$, one has $\|f-p\| \geqslant \|\tilde{T}_{m,n}\|$ for all $p \in Q_{m,n}$, where $\tilde{T}_{m,n}(w, z)=\tilde{T}_m(w)\,\tilde{T}_n(z)$.

Proof. As indicated above, one has

$$T_m(w_k) = (-1)^k \, \|T_m\|, \qquad \|T_m\| = \frac{\rho^m + \rho^{-m}}{2},$$

$$w_k = \frac{1}{2}\,(\rho e^{ik\pi/m} + \rho^{-1} e^{-ik\pi/m}), \qquad k = 0, 1, ..., 2m-1, \quad (19)$$

$$T_n(z_j) = (-1)^j \, \|T_n\|, \qquad \|T_n\| = \frac{\rho^n + \rho^{-n}}{2},$$

$$z_j = \frac{1}{2}\,(\rho e^{ij\pi/n} + \rho^{-1} e^{-ij\pi/n}), \qquad j = 0, 1, ..., 2n-1, \qquad (20)$$

together with

$$\sum_{k=0}^{2m-1} (-1)^k \, w_k^h = 0, \qquad 0 \leqslant h < m,$$

$$\sum_{j=0}^{2n-1} (-1)^j \, z_j^l = 0, \qquad 0 \leqslant l < n.$$

Therefore, one gets $T_{m,n}(w_k, z_j) = (-1)^{k+j} \|T_{m,n}\|$ with $\|T_{m,n}\| = \|T_m\| \, \|T_n\|$ and

$$\sum_{k=0}^{2m-1} \sum_{j=0}^{2n-1} (-1)^{k+j} \, w_k^h z_j^l = 0, \qquad \min(h-m, l-n) < 0.$$

The desired conclusion follows from Theorem 2, in which P_{m+n-1} is replaced by $Q_{m,n}$. ∎

As $w^m z^n - \tilde{T}_m(w) \, \tilde{T}_n(z)$ is an element of P_{m+n-1}, which is a subspace of $Q_{m,n}$, it also solves Problem B. In fact, it is the unique best approximation in conditions identical to those of the real case.

THEOREM 9. *The best approximation to $w^m z^n$, $m + n \geqslant 1$, $m \geqslant n$, on B_ρ, out of P_{m+n-1} is unique if $n = 0$ or $n = m$.*

Proof. In view of Corollary 1, all best errors

$$\tilde{T}_m(w) \, \tilde{T}_n(z) + p(w, z), \qquad p \in P_{m+n-1},$$

are such that $p(w_k, z_j) = 0$ where w_k $(0 \leqslant k \leqslant 2m-1)$ and z_j $(0 \leqslant j \leqslant 2n-1)$ are defined in (19) and (20).

For $n = 0$, one has $p(w_k, z) = 0$, $k = 0, 1, ..., 2m-1$, for all $z \in E_\rho$. From $p(w, z) = \sum_{i=0}^{m-1} a_i(w) \, z^i$ where a_i is a polynomial of degree $m - 1 - i$, one deduces $a_i(w_k)=0$, $k=0, 1, ..., 2m-1$; hence $a_i=0$ for $i=0, 1, ..., m-1$, i.e., $p=0$.

For $n \neq 0$, p must vanish at all intersection points of the two curves

$$\hat{u}(w, z)=\prod_{k=0}^{2m-1}(w-w_k)=2^{1-2m}\left[T_{2m}(w)-\frac{\rho^{2m}+\rho^{-2m}}{2}\right]=0,$$

$$\hat{v}(w, z)=\prod_{j=0}^{2n-1}(z-z_j)=2^{1-2n}\left[T_{2n}(z)-\frac{\rho^{2n}+\rho^{-2n}}{2}\right]=0.$$

Applying Noether's theorem stated in the complex field, we obtain $p=\tilde{u}\,u+\hat{v}\,v$, $u \in P_{n-m-1}$, $v \in P_{m-n-1}$. As $n \leqslant m$, this yields $u=0$ and the set of optimal errors

$$\tilde{T}_m(w)\,\tilde{T}_n(z)+2^{1-2n}\left[T_{2n}(z)-\frac{\rho^{2n}+\rho^{-2n}}{2}\right]v(w, z), \qquad v \in P_{m-n-1}, \tag{21}$$

which is a singleton for $m=n$. ∎

In contrast with the real problem, the polynomial v in (21) must obey stringent conditions in case of nonuniqueness. For further convenience, we set, for $k \in \mathbf{Z}$, $t_k=(\rho^k+\rho^{-k})/2$, $u_k=(\rho^k-\rho^{-k})/2$, and establish

THEOREM 10. *For $m>n \neq 0$, if the function $2^{2-m-n}f(w, z)$ with*

$$f(w, z)=T_m(w)\,T_n(z)+[T_{2n}(z)-t_{2n}]\,q(w, z) \tag{22}$$

is an optimal error related to Problem B, *for some polynomial q of degree $d \leqslant m-n-1$, then q has necessarily the form*

$$q(w, z)=\begin{cases}\alpha, & d<n,\\ \alpha+r(z), & n \leqslant d \leqslant 2n,\\ \alpha+r(z)+s(w, z), & d>2n,\end{cases} \tag{23}$$

where $\alpha \in \mathbf{R}$ and

$$\begin{aligned} r(z)=\alpha_0 T_n(z)&+\sum_{j=1}^{d-n}\alpha_j\frac{T_{n+j}(z)}{u_{n+j}}+\sum_{j=1, j \neq n}^{d-n}\alpha_j\frac{T_{|n-j|}(z)}{u_{n-j}}\\ &+i\sum_{j=1}^{d-n}\beta_j\left[\frac{T_{n+j}(z)}{t_{n+j}}-\frac{T_{|n-j|}(z)}{t_{n-j}}\right], \qquad \alpha_0, \alpha_j, \beta_j \in \mathbf{R}, \end{aligned} \tag{24}$$

$$s(w, z)=[T_{2n}(z)-t_{2n}]\,w\,t(w, z), \qquad t \in P_{d-2n-1}. \tag{25}$$

The following lemma is needed.

LEMMA 1. *In order that $2^{2-m-n}f(w, z)$ be an optimal error, q must be real-valued at all extreme points of $T_m(w)\,T_n(z)$ on B_ρ, i.e., $q(w_k, z_j) \in \mathbf{R}$, $0 \leqslant k \leqslant 2m-1$, $0 \leqslant j \leqslant 2n-1$, where w_k and z_j are defined in* (19) *and* (20).

Proof. Denoting the points (w, z) on B_ρ by

$$w = \frac{\rho e^{i\phi} + \rho^{-1} e^{-i\phi}}{2} = t_1 \cos\phi + i\,u_1 \sin\phi, \qquad 0 \leqslant \phi \leqslant 2\pi,$$

$$z = \frac{\rho e^{i\theta} + \rho^{-1} e^{-i\theta}}{2} = t_1 \cos\theta + i\,u_1 \sin\theta, \qquad 0 \leqslant \theta \leqslant 2\pi,$$

we get

$$T_m(w) = t_m \cos m\phi + i\,u_m \sin m\phi,$$
$$T_n(z) = t_n \cos n\theta + i\,u_n \sin n\theta,$$

and

$$T_{2n}(z) - t_{2n} = 2 \sin n\theta(-t_{2n} \sin n\theta + i\,u_{2n} \cos n\theta).$$

Therefore, if $q_1(\phi, \theta)$ and $q_2(\phi, \theta)$ are the real and imaginary parts of q on B_ρ, we obtain by straightforward computations the square modulus of f on B_ρ as

$$\begin{aligned} F(\phi, \theta) = {} & (t_m^2 - \sin^2 m\phi)(t_n^2 - \sin^2 n\theta) \\ & + 4 \sin n\theta[q_1(\phi, \theta)\,f_1(\phi, \theta) + q_2(\phi, \theta)\,f_2(\phi, \theta)] \\ & + 4[q_1^2(\phi, \theta) + q_2^2(\phi, \theta)] \sin^2 n\theta(t_{2n}^2 - \cos^2 n\theta), \end{aligned} \tag{26}$$

where

$$f_1(\phi, \theta) = -t_m t_n \cos m\phi \sin n\theta \cos n\theta + u_m u_n \sin m\phi(t_{2n} + \cos^2 n\theta),$$
$$f_2(\phi, \theta) = -t_m u_n \cos m\phi(t_{2n} + \cos^2 n\theta) - u_m t_n \sin m\phi \sin n\theta \cos n\theta.$$

Since the set of optimal errors is characterized by $F(\phi, \theta) \leqslant t_m^2 t_n^2$, all $(\phi, \theta) \in \mathbf{R}^2$, the points $(\phi_k = k\pi/m,\ \theta_j = j\pi/n)$, $k = 0, 1, ..., 2m-1$, $j = 0, 1, ..., 2n-1$, such that $F(\phi_k, \theta_j) = t_m^2 t_n^2$, must be stationary points of F. One easily verifies that $F_\phi(\phi_k, \theta_j) = 0$ and $F_\theta(\phi_k, \theta_j) = (-1)^{k+j+1}\, 8nt_m t_n^2 u_n q_2(\phi_k, \theta_j)$. Hence, a necessary condition for optimality is $q_2(\phi_k, \theta_j) = 0$. ∎

We now turn to the proof of Theorem 10.

Proof of Theorem 10. Considering first a univariate polynomial $q(z)$ of degree d, we expand it into a Chebyshev series

$$q(z)=\sum_{l=0}^{d}(a_l+i\,b_l)\,T_l(z), \qquad a_l, b_l \in \mathbf{R},$$

to get, for $z=t_1\cos\theta+i\,u_1\sin\theta$,

$$\operatorname{Im} q(z)=F(\theta)=\sum_{l=0}^{d} b_l t_l \cos l\theta+\sum_{l=1}^{d} a_l u_l \sin l\theta; \tag{27}$$

that is, a trigonometric polynomial of order d, which must vanish at $\theta_j=j\pi/n$, $j=0, 1, ..., 2n-1$. In case of $d<n$, $F(\theta)$ is thus identically zero, which gives $a_1=\cdots=a_d=b_0=\cdots=b_d=0$ so that $q(z)$ is a real constant. For $d\geqslant n$, it is easy to see that $F(\theta)$ must contain the factor $\sin n\theta$

$$F(\theta)=\sin n\theta\left(\sum_{h=0}^{d-n} c_h\cos h\theta+\sum_{h=1}^{d-n} d_h \sin h\theta\right),$$

which, by simple trigonometric transformations, becomes

$$F(\theta)=c_0\sin n\theta+\frac{1}{2}\sum_{h=1}^{d-n}\{c_h[\sin(n+h)\theta+\sin(n-h)\theta]$$
$$+d_h[\cos(n-h)\theta-\cos(n+h)\theta]\}. \tag{28}$$

Finally, identifying (27) and (28) shows that q may be expressed according to (24).

If q is a bivariate polynomial of degree d, one has $q(w,z)=\sum_{l=0}^{d}A_l(z)\,w^l$ where A_l is of degree $d-l$ in z, such that $\operatorname{Im} q(w_k,z_j)=0$ for $k=0, 1, ..., 2m-1$ and $j=0, 1, ..., 2n-1$. As $d\leqslant m-n-1<m$, by the foregoing argument, the $2n$ univariate polynomials of degree d, $q_j(w)=q(w,z_j)$ $(0\leqslant j\leqslant 2n-1)$, that obey $\operatorname{Im} q_j(w_k)=0$, $0\leqslant k\leqslant 2m-1$, must be real constants. Hence, we obtain for $l=1, 2, ..., d$, $A_l(z_j)=0$, $0\leqslant j\leqslant 2n-1$, i.e., $A_l(z)=[T_{2n}(z)-t_{2n}]\,B_l(z)$, where the polynomial B_l has degree $d-l-2n$. To conclude, q is a univariate polynomial in z for $d\leqslant 2n$, whereas it is the sum of this polynomial and the bivariate polynomial (25) for $d>2n$. ▮

It is very hard to obtain precise bounds on the norm of the various admissible polynomials q, mentioned in Theorem 10. Yet, when $q(w,z)=a\in\mathbf{R}$, we get the following

THEOREM 11. *For $m>n\neq 0$, the function $2^{2-m-n}f(w,z)$ with*

$$f(w,z)=T_m(w)\,T_n(z)+\alpha[T_{2n}(z)-t_{2n}], \qquad \alpha\in\mathbf{R}, \tag{29}$$

is an optimal error for Problem B, *iff*

$$|\alpha| \leqslant A = \frac{1}{2}\frac{t_m}{t_n}\left[1+(1+4t_m^2u_n^2)^{1/2}\right]^{-1}. \tag{30}$$

Proof. Putting $q_1 = \alpha$, $q_2 = 0$ in (26) yields the square modulus of f as

$$\begin{aligned} F(\phi, \theta) = {} & (t_m^2 - \sin^2 m\phi)(t_n^2 - \sin^2 n\theta) \\ & + 4\,\alpha \sin n\theta[-t_m t_n \cos m\phi \sin n\theta \cos n\theta \\ & + u_m u_n \sin m\phi (t_{2n} + \cos^2 n\theta)] \\ & + 4\,\alpha^2 \sin^2 n\theta (t_{2n}^2 - \cos^2 n\theta), \end{aligned} \tag{31}$$

so that the best errors are characterized by $F(\phi, \theta) \leqslant t_m^2 t_n^2$ for all $(\phi, \theta) \in \mathbf{R}^2$.
By the proof of Lemma 1, we know that, at $(\phi_k, \theta_j) = (k\pi/m, j\pi/n)$, $F(\phi_k, \theta_j) = t_m^2 t_n^2$ and $F_\phi(\phi_k, \theta_j) = F_\theta(\phi_k, \theta_j) = 0$. Hence, $F(\phi_k, \theta_j)$ will be a local maximum iff the Hessian matrix of F is negative definite at (ϕ_k, θ_j). We compute $F_{\phi\phi}(\phi_k, \theta_j) = -2m^2 t_n^2 < 0$ and the determinant of the Hessian matrix at (ϕ_k, θ_j)

$$4\,m^2 n^2 t_m^2 t_n^2 \left[1 + (-1)^{k+j}\, 4\,\alpha \frac{t_n}{t_m} - 4\,\alpha^2 u_{2n}^2\right], \tag{32}$$

or

$$\begin{aligned} & 16\,m^2 n^2 t_m^2 t_n^2 u_{2n}^2 \left\{\frac{1}{2}\frac{t_m}{t_n}\left[(-1)^{k+j} + (1+4\,t_m^2 u_n^2)^{1/2}\right]^{-1} + \alpha\right\} \\ & \qquad \left\{\frac{1}{2}\frac{t_m}{t_n}\left[-(-1)^{k+j} + (1+4\,t_m^2 u_n^2)^{1/2}\right]^{-1} - \alpha\right\}, \end{aligned}$$

which is nonnegative for all (k, j) iff condition (30) is satisfied.

It remains to prove that the points (ϕ_k, θ_j) are also global maximum points of F for $|\alpha| \leqslant A$, i.e., $G(\phi, \theta) = t_m^2 t_n^2 - F(\phi, \theta) \geqslant 0$, for all $(\phi, \theta) \in \mathbf{R}^2$ and $|\alpha| \leqslant A$. Due to the apparent symmetry of G, it suffices to consider the values $\pi/2 \leqslant m\phi \leqslant \pi$, $0 \leqslant n\theta \leqslant \pi/2$ or, introducing the variables $\lambda = m\phi - \pi/2$, $\sigma = n\theta$, the square $[0, \pi/2]^2$, such that G becomes

$$\begin{aligned} g(\lambda, \sigma) = {} & t_n^2 \sin^2 \lambda + t_m^2 \sin^2 \sigma - \sin^2 \lambda \sin^2 \sigma \\ & - 4\,\alpha \sin \sigma[t_m t_n \cos \lambda \cos \sigma \sin \sigma + u_m u_n \sin \lambda (2t_n^2 - \sin^2 \sigma)] \\ & - 4\,\alpha^2 \sin^2 \sigma (u_{2n}^2 + \sin^2 \sigma). \end{aligned}$$

Clearly, one has $g(\lambda, \sigma) \geqslant 0$, for all $|\alpha| \leqslant A$, iff $g(\lambda, \sigma)$ is nonnegative for $\alpha = A$. Therefore, putting $\alpha = A$ in g and using, by (3.2), the relation

$$1 - 4A \frac{t_n}{t_m} - 4A^2 u_{2n}^2 = 0,$$

we obtain, by direct computations, the decomposition $g = g_1 + g_2$, where g_1 is the nonnegative function

$$g_1(\lambda, \sigma) = \left[t_n(\sin \lambda - 4Au_m u_n \sin \sigma) - \frac{1}{2t_n} \sin \lambda \sin^2 \sigma \right]^2,$$

whereas $g_2(\lambda, \sigma) = \sin^2 \sigma \, h(\lambda, \sigma)$ such that the function h written in the variables $u = \cos \lambda$, $v = \cos \sigma$, is

$$\begin{aligned} h(\lambda, \sigma) = H(u, v) &= 4At_m t_n(1 - u\, v) - 4A^2(1 - v^2) \\ &\quad - \frac{1}{4t_n^2}(1 - u^2)(1 - v^2). \end{aligned} \tag{33}$$

The proof is this completed if one has $H(u, v) \geqslant 0$ for all $(u, v) \in [0, 1]^2$. On the edges of the square $[0, 1]^2$, one finds $H(u, 1) = 4At_m t_n(1 - u) \geqslant 0$, $H(1, v) = 4A(1 - v)[t_m t_n - A(1 + v)] \geqslant 4A(1 - v)(t_m t_n - 2A) \geqslant 0$, $H(u, 0) \geqslant H(0, 0)$, $H(0, v) \geqslant H(0, 0)$ and it is shown in Appendix that $H(0, 0) > 0$. On the other hand, the stationary points of H are solutions of

$$H_u = -4At_m t_n v + \frac{1}{2t_n^2} u(1 - v^2) = 0,$$

$$H_v = -4At_m t_n u + 8A^2 v + \frac{1}{2t_n^2} v(1 - u^2) = 0.$$

Eliminating u from both equations, we get

$$v \left[(1 - v^2)^2 - \frac{4A^2 t_m^2 t_n^4}{A^2 + (16t_n^2)^{-1}} \right] = 0.$$

The first factor of the left side yields $v = 0$, hence $u = 0$. The second factor cannot vanish for $0 \leqslant v \leqslant 1$, because one has $(1 - v^2)^2 \leqslant 1$, and, as shown in Appendix,

$$4A^2 t_m^2 t_n^4 > A^2 + \frac{1}{16t_n^2}. \tag{34}$$

Consequently, there is no interior stationary point of H in $[0, 1]^2$ and the theorem is established. ∎

Combining Theorems 10 and 11, we conclude that (29) *and* (30) *characterize the set of optimal errors for Problem* B *in case of* $2n+1>m>n>0$. For $m \geqslant 2n+1>1$, there exist other forms of optimal errors. For instance, if the polynomial q in (22) is given by $\alpha_0 T_n(z)$, $\alpha_0 \in \mathbf{R}$, it can be shown by a lengthy proof, similar to that of Theorem 11, that it corresponds to best errors iff $|\alpha_0| \leqslant (t_m/2t_n^2)[t_{2n}+(t_{2n}^2+4t_m^2u_n^2)^{1/2}]^{-1}$.

As $\rho \to 1$, the biellipse B_ρ collapses to the unit square $U=[-1,+1]^2$ and, by (30), one has $A \to \frac{1}{4}$ in accordance with (14). As $\rho \to \infty$, the difference between the semi-axes of E_ρ tends to zero. In fact, on the bidisk, like in the univariate case [2, p. 146], we can prove by an argument based on the maximum modulus theorem: 0 *is the unique best uniform approximation to* $w^m z^n$, *out of* P_{m+n-1}, *on the bidisk* $\{(w,z) \in \mathbf{C}^2, |w| \leqslant R, |z| \leqslant R, R>0\}$. Except for the uniqueness, this is also a consequence of a result by Reimer [11].

APPENDIX: PROOF OF TWO INEQUALITIES

1. $H(0,0)>0$. From (33), $H(0,0)$ is given by

$$H(0,0)=4At_m t_n - 4A^2 - \frac{1}{4t_n^2}, \tag{35}$$

and can be factorized as

$$H(0,0)=\left[\left(t_m^2 t_n^2 - \frac{1}{4t_n^2}\right)^{1/2} + t_m t_n - 2A\right]\left[\left(t_m^2 t_n^2 - \frac{1}{4t_n^2}\right)^{1/2} - t_m t_n + 2A\right],$$

in which the first factor is positive since $t_m t_n > 2A$. In order to check the positivity of the second factor

$$2A > t_m t_n - \left(t_m^2 t_n^2 - \frac{1}{4t_n^2}\right)^{1/2} = \frac{1}{4t_n^2}\left[t_m t_n + \left(t_m^2 t_n^2 - \frac{1}{4t_n^2}\right)^{1/2}\right]^{-1},$$

we use definition (30) of A to get the inequality

$$1+(1+4t_m^2u_n^2)^{1/2} < 4t_m^2 t_n^2 + (16t_m^4 t_n^4 - 4t_m^2)^{1/2},$$

which is satisfied if $1+4t_m^2u_n^2 < 16t_m^4t_n^4 - 4t_m^2$ or, by $u_n^2 = t_n^2 - 1$, if $4t_m^2 t_n^2(2t_m^2 t_n^2 - 1) + (8t_m^4 t_n^4 - 1) > 0$. The last inequality is an evidence since the left side is the sum of two positive terms.

2. *Inequality* (34). Since $H(0,0)>0$, (35) yields $At_m t_n > A^2 + 1/16t_n^2$. Therefore, inequality (34) is true if $4A^2t_m^2t_n^4 \geqslant At_m t_n$ or $(4At_n/t_m)\, t_m^2 t_n^2 \geqslant 1$. By definition (30) of A, we get $2t_m^2 t_n^2 - 1 \geqslant (1+4t_m^2u_n^2)^{1/2}$ and, squaring both members, $t_m^4 t_n^4 - t_m^2 t_n^2 \geqslant t_m^2 u_n^2$. Using the relations $u_n^2 = t_n^2 - 1$, $u_m^2 = t_m^2 - 1$, we finally obtain $u_m^2 t_n^4 + (t_n^2 - 1)^2 \geqslant 0$.

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